

The Usage of the Polar Coordinates

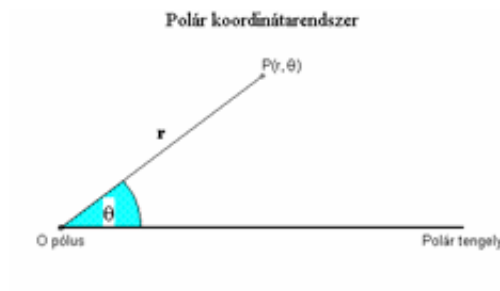
Prove that the

$$r=2\sin(\theta);$$

equation given in the polar coordinates in the plane determines a circle then draw the circle.

The arbitrary P point of the plane is identified with the $[r, \theta]$ real pair of numbers in the polar coordinate system. A half-line with an O start point is needed for the identification. The O start point of the half-line is called the pole and the half-line is called the polar axis. The number r represents the distance between the O and P points while the θ measures the rotation angle between the polar axis and the OP half-line in radian. During the rotation, the polar axis is rotated into the OP half-line. If we do the rotation counter clockwise then the θ is positive, otherwise it is negative.

These basic concepts are illustrated by the figure below.



well-known relations. We can switch from the $[r, \theta]$ variables to the $[x, y]$ variables. However, this is the other way round in our task: the expressions of the x and y have to be substituted to the places of the r and θ . If we solve the system of equation above for the r and the θ we get the following transitional formula:

$$x = r \cos(\theta), y = r \sin(\theta)$$

ismert összefüggéseket kapjuk. Ezek segítségével térhetünk át az $[r, \theta]$ változóról az $[x, y]$ változókra. A kitöltött feladatban azonban éppen fordítva, az r és θ helyére kell helyettesíteni x és y kifejezéseit. Ha a fenti egyenletrendszer r-re és θ -ra megoldjuk, akkor megkapjuk a következő áttérési képleteket:

$$r = \sqrt{x^2 + y^2}, \theta = \arctan\left(\frac{y}{x}\right).$$

We could start to solve the task at this point but before that we are going to deduce the transitional formulas between the right angle triangle and the polar coordinate system in Maple. The following two deductions will not only broaden your mind about the functions and expressions but it will also help you to acquire useful manipulation techniques.

Enter the equations concerning the x and y!

```
[
> restart
> x = r cos(theta)
```

$$x = r \cos(\theta)$$

(1)

$$\begin{aligned} > y = r \sin(\theta) \\ & \qquad \qquad \qquad y = r \sin(\theta) \end{aligned} \tag{2}$$

We have to get the values of the r and the θ from the system of equation. The first thing that crosses our mind is to solve the (1,2) system of equation for the r and θ unknowns. This can be easily done in a computer-algebra system.

$$\begin{aligned} > \text{solve}(\{x = r \cos(\theta), y = r \sin(\theta)\}, \{r, \theta\}) \\ \left\{ r = \text{RootOf}(-y^2 - x^2 + _Z^2), \theta = \arctan\left(\frac{y}{\text{RootOf}(-y^2 - x^2 + _Z^2)}, \right. \right. \\ \left. \left. \frac{x}{\text{RootOf}(-y^2 - x^2 + _Z^2)}\right) \right\} \end{aligned} \tag{3}$$

The RootOf notation has come up again. We also got the equations the roots of which can be used to solve the task. Let's consider only the first solution out of the others and simplify the expressions. By the way, do you check the calculations of Maple or you believe that the solutions are correct?

$$\begin{aligned} > \text{allvalues}(\%) \\ \left\{ r = \sqrt{y^2 + x^2}, \theta = \arctan\left(\frac{y}{\sqrt{y^2 + x^2}}, \frac{x}{\sqrt{y^2 + x^2}}\right) \right\}, \left\{ r = -\sqrt{y^2 + x^2}, \theta = \arctan\left(\right. \right. \\ \left. \left. -\frac{y}{\sqrt{y^2 + x^2}}, -\frac{x}{\sqrt{y^2 + x^2}} \right) \right\} \end{aligned} \tag{4}$$

$$\begin{aligned} > \%_1 \\ \left\{ r = \sqrt{y^2 + x^2}, \theta = \arctan\left(\frac{y}{\sqrt{y^2 + x^2}}, \frac{x}{\sqrt{y^2 + x^2}}\right) \right\} \end{aligned} \tag{5}$$

$$\begin{aligned} > \text{attérés}_1 := \left\{ r = \sqrt{y^2 + x^2}, \theta = \arctan\left(\frac{y}{\sqrt{y^2 + x^2}}, \frac{x}{\sqrt{y^2 + x^2}}\right) \right\}_1 \\ \text{attérés}_1 := r = \sqrt{y^2 + x^2} \end{aligned} \tag{6}$$

$$\begin{aligned} > \text{attérés}_2 := \left\{ r = \sqrt{y^2 + x^2}, \theta = \arctan\left(\frac{y}{\sqrt{y^2 + x^2}}, \frac{x}{\sqrt{y^2 + x^2}}\right) \right\}_2 \\ \text{attérés}_2 := \theta = \arctan\left(\frac{y}{\sqrt{y^2 + x^2}}, \frac{x}{\sqrt{y^2 + x^2}}\right) \end{aligned} \tag{7}$$

$$\begin{aligned} > \text{map}(\tan, \%) \\ \tan(\theta) = \frac{y}{x} \end{aligned} \tag{8}$$

We used a Maple procedure, called map. Its call is

$$\text{map}(\text{name of the function}, \text{expression})$$

The map procedure considers the function appearing in its first argument and applies it for all the components of the expression. Then it creates a Maple object which has the same construction as the original expression from the objects created. In our case this expression is the equation. So the map considered the components of the equation, namely the left and right sides of the equation and it applied

the tangent function for these then it created a new equation from the expressions received.

As you can see in the case of such an easy issue the formulas and the steps to be taken can become very complex after using the solve procedure. It has become more difficult to solve the problem this way than by solving it on paper and in pencil.

Let's approach the deduction of the transitional formulas from another viewpoint. Solve the problem as we would have done it on paper. Square both parts of the (1,2) equations then add them. After the simplifications, we extract the square roots of both sides of the equation received, supposing that the $0 < r$.

$$\begin{aligned} > \text{map}(x \rightarrow x^2, x = r \cos(\theta)) + \text{map}(x \rightarrow x^2, y = r \sin(\theta)) \\ & \qquad \qquad \qquad x^2 + y^2 = r^2 \cos^2(\theta) + r^2 \sin^2(\theta) \end{aligned} \quad (9)$$

$$\begin{aligned} > \text{simplify}(\%) \\ & \qquad \qquad \qquad x^2 + y^2 = r^2 \end{aligned} \quad (10)$$

$$\begin{aligned} > \text{map}(\text{sqrt}, \%) \\ & \qquad \qquad \qquad \sqrt{x^2 + y^2} = \sqrt{r^2} \end{aligned} \quad (11)$$

$$\begin{aligned} > \text{simplify}(\%, \text{assume} = \text{positive}) \\ & \qquad \qquad \qquad \sqrt{x^2 + y^2} = r \end{aligned} \quad (12)$$

We have received the first relation.

To get the expression concerning the y , divide the two equations by each other. For this, the left and right sides of the equations have to be divided by each other.

$$\begin{aligned} > \frac{\text{lhs}(\text{(2)})}{\text{lhs}(\text{(1)})} = \frac{\text{rhs}(\text{(2)})}{\text{rhs}(\text{(1)})} \\ & \qquad \qquad \qquad \frac{y}{x} = \frac{\sin(\theta)}{\cos(\theta)} \end{aligned} \quad (13)$$

We have received the formula we have been searching for such a long time. Anyway, we can consider this formula the transitional formula because – as we will see – it is more useful than the original one.

However, we should keep on trying to get the original syntax of the transitional formula. First, apply the arctangent function for both sides of the equation received.

$$\begin{aligned} > \text{map}(\text{arctan}, \%); \\ & \qquad \qquad \qquad \arctan\left(\frac{y}{x}\right) = \arctan\left(\frac{\sin(\theta)}{\cos(\theta)}\right) \end{aligned} \quad (14)$$

We would expect that after the $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$ substitution and simplification we are going to get the second transitional formula.

$$\begin{aligned} > \text{student}_{\text{powsubs}}\left(\frac{\sin(\theta)}{\cos(\theta)} = \tan(\theta), \%\right) \\ & \qquad \qquad \qquad \arctan\left(\frac{y}{x}\right) = \arctan(\tan(\theta)) \end{aligned} \quad (15)$$

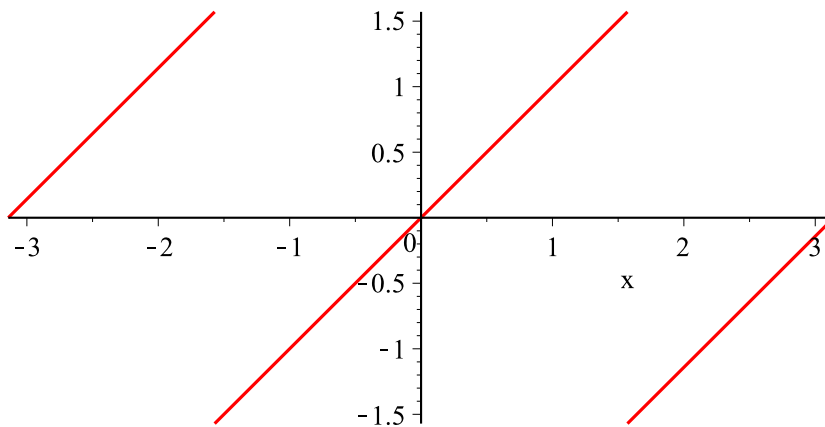
> *simplify*((15))

$$\arctan\left(\frac{y}{x}\right) = \arctan(\tan(\theta)) \quad (16)$$

Well, we did not expect this because we should have got the θ on the right side. Why did the composition of the inverse and original functions in the $\arctan(\tan(\theta))$ expression not give the θ automatically? In these special cases, use the plot procedure. Let's draw the $y=\arctan(\tan(x))$ function and check if its graph coincides with the $y=x$ bisector line.

> *plot*(*arctan*(*tan*(*x*)), *x* = $-\pi..pi$, *discont* = *true*, *title* = "*y=arctan*(*tan*(*x*))", *scaling* = *constrained*)

y=arctan(*tan*(*x*))



Maple was right again. We expected that the composition of the inverse and original function would give the identical mapping. Naturally it is true but not for all the x -es. We can see in the graph that the $\arctan(\tan(x))=x$ equation is true in the $-\frac{\pi}{2} < x < \frac{\pi}{2}$ interval. However, this is not true for all the x -es thus the automatic simplify procedure showed the composition correctly. Notice that another option of the plot procedure, the *discont* was also used during the process. Since the $y=\arctan(\tan(x))$ function has a split at the $x = \frac{\pi}{2} + k \cdot \pi$ point, the *discont=true* equality indicates to the plot that it should not connect the calculated function values near the split points. Thus the splits really look like splits. In the case of the default *discont=false*, the plot connects the jump with a straight line at the splits so the

functions will be similar to the saw teeth.

As we are now dealing with simplifications, let's explore the settings of the simplify instruction. We can make assumptions about the variables of the expression to be simplified in the same identification line by the simplify instruction subsequent to the assuming keyword.

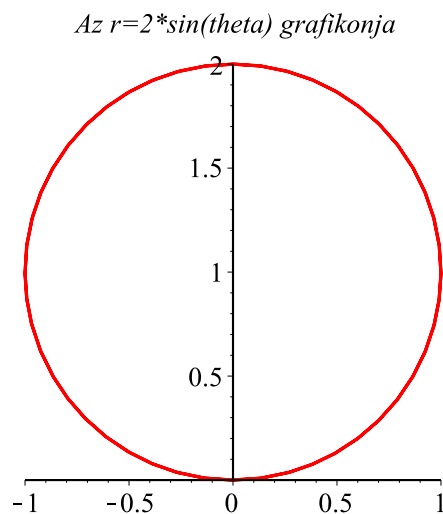
```
> simplify( (15) ) assuming  $-\frac{\pi}{2} < \theta$  and  $\theta < \frac{\pi}{2}$   
arctan( $\frac{y}{x}$ ) =  $\theta$  (17)
```

The instruction above simplifies the expressions of the (15) equation concerning the $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ assumption. Under these circumstances the $\arctan(\tan()) = \theta$ equation is going to be true and Maple also responded this.

We have thoroughly discussed the transitional formulas so now we can start to solve the task by drawing a circle. So far we have only drawn in the Cartesian coordinate system with the plot procedure. However, the plot procedure can also be used to draw curves given in the polar coordinates by entering the `coords=polar` option. Create the `r` function and draw the graph of the curve we are looking for.

```
> r := 2 sin(theta)  
2 sin(theta) (18)
```

```
> plot([r, theta, theta = 0 .. 2 Pi], coords = polar, title = Az r=2*sin(theta) grafikonja, scaling = CONSTRAINED)
```



We have finished the graphical part of the task.

But before we continue, note the syntax of the plot procedure in the case of drawing in the polar coordinates:

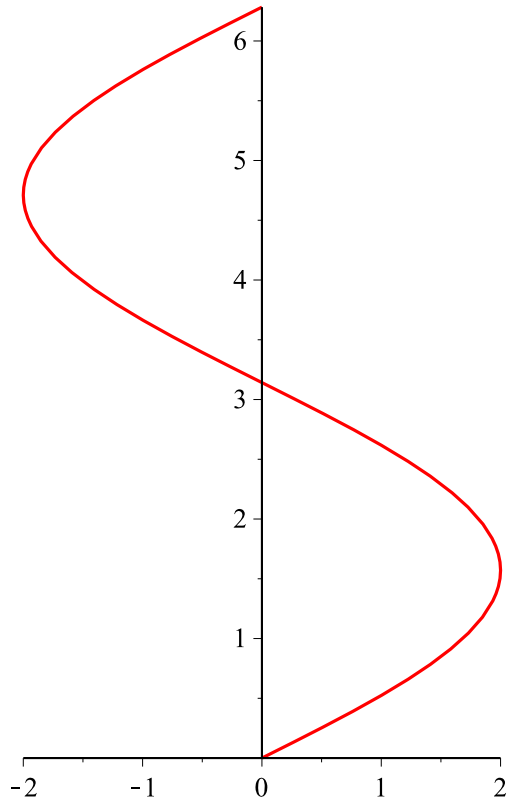
```
plot([r(theta), theta, theta=a..b], coords=polar, futher option).
```

The first parameter of the plot procedure is a three-element list. The first element of the list is an expression of the `r`, the second is the variable of the expression (namely the `theta`) and the third describes

the value domain of the angle. The `cords=polar` is necessary out of the other options of the plot procedure for drawing in the polar coordinate systems.

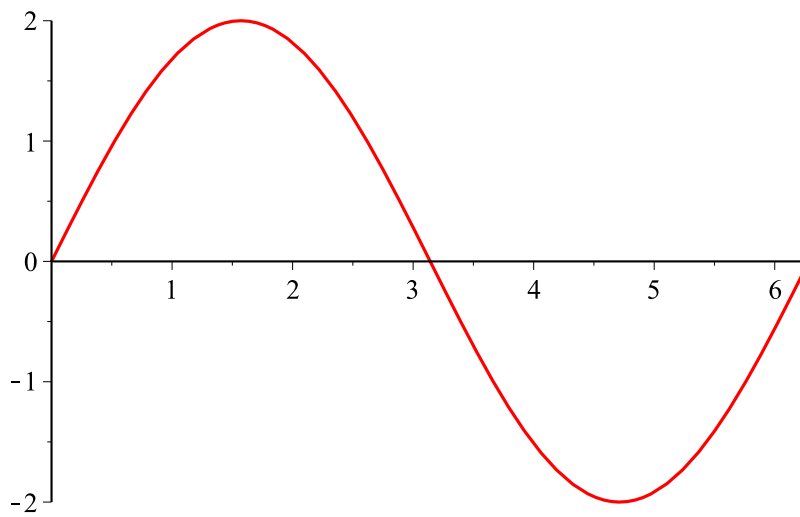
What would be the result of the drawing if we omit the `cords=polar` option from the previous plot instruction? Here you can see the instruction and its result:

```
> plot([r, theta, theta = 0 ..2 * Pi], scaling = CONSTRAINED);
```



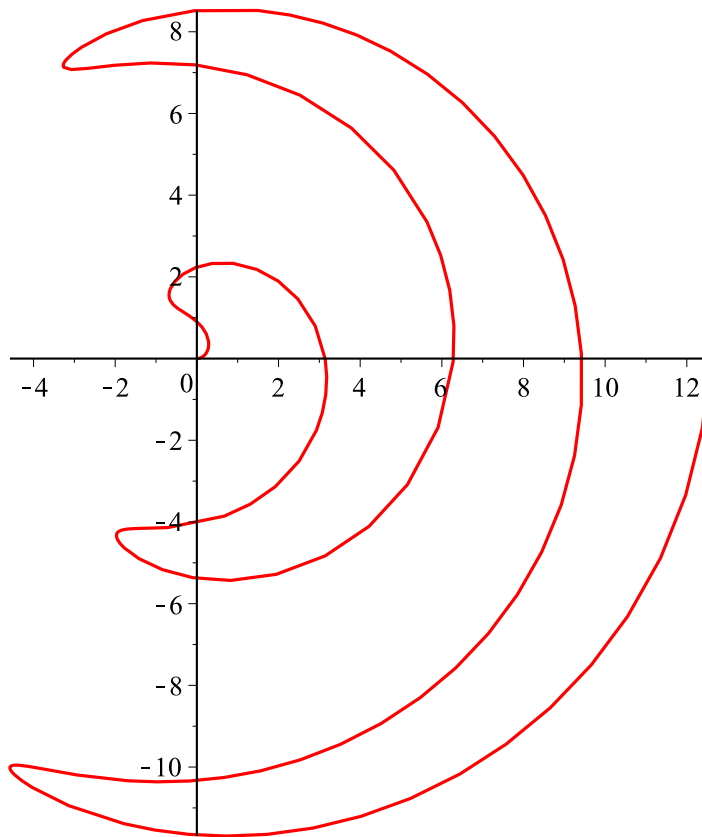
This is a sine type curve which waves at the y axis because we mirrored the part above the $[0, 2\pi]$ interval of the graph of the $y=2\sin(x)$ function to the $y=x$ bisector line and drew the parametric curve given in the syntax of $[2 \sin(\theta), \theta, \theta = 0 ..2 \pi]$ We can get the parametric syntax of the graph of the $y=2\sin(x)$ function by transposing the x and the y, like this: $[\theta, 2 \sin(\theta), \theta = 0 ..2 \pi]$.

```
> plot([theta, 2 * sin(theta), theta = 0 ..2 * Pi], scaling = CONSTRAINED);
```



We can get an interesting curve if we draw the trigonometric function in the polar coordinate system again.

```
> plot([theta, 2 * sin(theta), theta = 0 .. 4 * Pi], coords = polar, scaling = CONSTRAINED);
```



The graph coinciding with the $r=2\sin(\theta)$ polar equation can be drawn with the polarplot procedure of the plots package.

```
> plots_polarplot(r, scaling = constrained) :
```

We are not going to show the output of the instruction since it coincides with the circle drawn above. The syntax of the polarplot located in the plots package is

```
polarplot(r(theta), theta = a .. b, további opciók)
```

or

```
polarplot([r(t), theta(t), t = a .. b], további opciók)
```

The polarplot has all those properties which the plot has concerning the cords=polar assumption.

Let's look at the first part of the task. Prove that the curve received is a circle. It is known that the equation of an arbitrary circle in a right angle coordinate system is

$$(x - u)^2 + (y - v)^2 = R^2$$

in which case the R is the radius of the circle and the $[u,v]$ are the centre of the circle. For the proof, we have to convert the $r=2\sin(\theta)$ equation into an equation given in x and y right angle coordinates with the help of the transitional formula.

> restart

> $\text{áttérés}_1 := r = \sqrt{x^2 + y^2}$

$$\text{áttérés}_1 := r = \sqrt{x^2 + y^2} \quad (19)$$

> $\text{áttérés}_2 := \theta = \arctan\left(\frac{y}{x}\right)$

$$\text{áttérés}_2 := \theta = \arctan\left(\frac{y}{x}\right) \quad (20)$$

> $\text{kör} := r = 2 \sin(\theta)$

$$\text{kör} := r = 2 \sin(\theta) \quad (21)$$

> $\text{subs}(\{\text{áttérés}_1, \text{áttérés}_2\}, \text{kör})$

$$\sqrt{x^2 + y^2} = 2 \sin\left(\arctan\left(\frac{y}{x}\right)\right) \quad (22)$$

Simplify the trigonometric expression of the equation then create the x^2+y^2 expression from the equation with the isolate procedure.

> $\text{simplify}(\%)$

$$\sqrt{x^2 + y^2} = \frac{2y}{\sqrt{\frac{x^2 + y^2}{x^2}} x} \quad (23)$$

> $\text{map}(t \rightarrow t^2, \%)$

$$x^2 + y^2 = \frac{4y^2}{x^2 + y^2} \quad (24)$$

> $\text{isolate}(\%, x^2 + y^2)$

$$x^2 + y^2 = -2y \quad (25)$$

We have met a new procedure. The $\text{isolate}(\text{equation}, \text{sub-expression})$ call of the isolate procedure returns an equation which is equivalent to the given equation. However, the sub-expression comes on the left side of the new expression. Be aware that Maple cannot do this in every case.

Convert the syntax sorted to zero of the (25) equation to a complete square concerning the y .

> $\text{lhs}(\text{(25)}) - \text{rhs}(\text{(25)}) = 0$

$$x^2 + y^2 + 2y = 0 \quad (26)$$

> $\text{student}_{\text{completesquare}}(\%, y)$

$$(y + 1)^2 - 1 + x^2 = 0 \quad (27)$$

> $\text{map}(x \rightarrow x + 1, \text{(27)})$

$$(y + 1)^2 + x^2 = 1 \quad (28)$$

The sorting to zero was done by extracting the right side from the left side of the (25) equation. The

completesquare procedure of the second command, which can be found in the student package, is used to convert an expression into a complete square. Its syntax is:

completesquare(egyenlet, változó név);

It makes the equation be converted to a complete square concerning the given variable. Finally, by adding 1 to each side of the circle equation we can get the equation of a circle the centre of which is [0, 1] and its radius is 1. The operation of the previous map instruction should be mentioned as well. According to it, if we use the [képlet] function for the (27) equation, at first the x will be the left then the right side of the equation and add 1. Finally, it concatenates the two expressions with an equality sign. In other words, 1 was given to each side of the equation.

We have solved the task.

What Have You Learnt About Maple?

- With the help of these commands

plot([r(theta), theta, theta = a ..b], coords = polar, további opciók)

plots_polarplot(r(theta), theta = a ..b, további opciók)

we can draw a curve given in the polar coordinates. The curve is described by the $r:=\theta \rightarrow$ expression function. Maple calculates the given θ and $r(\theta)$ values then it determines the right angle coordinates according to the transitional formulas. We can get the points of the curve by the changing of the θ .

- With the help of these two commands

plot([r(t), theta(t), t = a ..b], coords = polar, további opciók)

plots_polarplot([r(t), theta(t), t = a ..b], további opciók)

we can draw a curve given parametric in the polar coordinates. The curve is determined by the $r=r(t)$, $\theta = \theta(t)$ parametric system of equation and the $t=a..b$; is the value domain of the t parameter of the representation.

- We have become familiar with two new methods used for the converting of the equations. The

map(függvény, objektum)

instruction applies the function for the components of the object. The object is an arbitrary Maple object thus it can be an equation as well. The type of the result coincides with the type of the object.

The $[egyenlet_1 + egyenlet_2]$ instruction adds the given equations. More precisely, it adds the left sides of the two equations and this will be the left side of the sum equation. Furthermore, it adds the right sides of the two equations and this will be the right side of the sum equation.

- The

simplify(kifejezés, assume = positive)

instruction simplifies the expression supposing that the its variables are positive.

- The $F@G$ Maple object is the complex function of the F and G functions.

•The *simplify(kifejezés, atsign)* instruction simplifies the expressions in which the $@$ operator of the complex function creation appears.

- During the conversion of the equations the

isolate(egyenlet, kifejezés)

instruction can be used which expresses the given sub-expression from the equation and it sorts it to the left side of the equation. We can draw a parallel between the isolate and the solve procedures.

•The call of the completesquare procedure

completesquare(kifejezés, változó)

located in the student package converts the expression into a complete square regarding the given variable.

Exercises

1. Deduct from the $r = 2 \sin(\theta)$ polar equation the equation of the circle by multiplying the equation by r and using the transitional formulas.

2. Prove with Maple that the $\tan(\arctan(x)) = x$ is fulfilled for all real x -es.

3. Try out the following Maple instruction: `> arctan(tan(x))-(arctan@tan)(x)`

Explain the result.

4. What is the difference between the $(\arctan@tan)(x)$ and the $\arctan@tan(x)$ expressions? Explain the result.

5. You can get spectacular graphs by drawing the graphs of the curves given in the following polar coordinates.

$r = 5$ $r = 2 + 2 \cos(\theta)$ $r = 2 \sin(2 \theta)$ $r = \theta$	$\theta = \frac{\text{Pi}}{2}$ $r = 4*(1-\sin(\theta))$ $r = 2 \sin(4 \theta)$ $r = \tan(\theta)$	$r = 4 \cos(\theta)$ $r = 1 + 2 \cos(\theta)$ $r = 8 \cos(5 \theta)$ $r = \frac{-2}{\sin(\theta)}$
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